Fingerprints of classical instability in open quantum dynamics

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The dynamics near a hyperbolic fixed point in phase space is modeled by an inverted harmonic oscillator. We investigate the effect of the classical instability on the open quantum dynamics of the oscillator, introduced through the interaction with a thermal bath, using both the survival probability function and the rate of von Neumann entropy increase, for large times. In this parameter range we prove, using influence functional techniques, that the survival probability function decreases exponentially at a rate κ' depending not only on the measure of instability in the model but also on the strength of interaction with the environment. We also show that κ' determines the rate of the von Neumann entropy increase and that this result is independent of the temperature of the environment. This generalizes earlier results that are valid in the limit of vanishing dissipation. The validity of inferring similar rates of survival probability decrease and entropy increase for quantum chaotic systems is also discussed. [S1063-651X(98)00610-2]

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I. INTRODUCTION

Quantum-classical correspondence for the case of classically chaotic systems has been much investigated recently (see, e.g., Refs. [1–6]). Expectation values of corresponding dynamical variables begin to differ [7,8], as do classical and quantum phase space distributions [9,10], on extremely short time scales that are typically logarithmic in Planck's constant. If these studies are taken at face value, therefore, all chaotic dynamical systems, being fundamentally quantum in nature, should either be obeying quantum laws of evolution now or be expected to do so in an extremely short time. Observations tell us otherwise, however.

Sarkar and Satchell [11], a decade ago, already pointed out the possible role of environment in the quantum evolution of chaotic systems. Recently [12], Zurek and Paz have conjectured an interesting quantitative relation between a classical chaotic system and its quantum version that is in contact with a bath. They have considered the Wigner representation of the quantum Liouville equation

$$\dot{W}=\{H,W\}_{PB}+\sum_{n=1}^{\infty}\frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}V}{\partial x^{2n+1}}\frac{\partial^{2n+1}W}{\partial p^{2n+1}} \quad (1)$$

for a particle in a potential $V(x)$ moving in a two-dimensional phase space. Clearly, the $\hbar$ terms are a singular perturbation of the classical Liouville equation, in that the order of the differential equation is changed. For chaotic systems derivatives of the Wigner function with respect to momentum become large enough to render the quantum correction terms comparable in magnitude to the classical Poisson bracket after the *Ehrenfest time*, $\tau_\hbar \propto (1/\lambda)\ln(1/\hbar)$, where λ is a Lyapunov exponent. Therefore, for $t>\tau_\hbar$ we expect significant differences between the classical and quantum descriptions of the same system.

We will now consider an environment consisting of harmonic oscillators to be coupled to the particle. The Caldeira-Leggett model [13,14] will be used for these oscillators. For the special case of a high-temperature, Ohmic environment the right-hand side of Eq. (1) is modified by the addition of a dissipitive and a decoherence term [12]. When the temperature is high enough and the dissipation sufficiently low the dissipative term may be considered unimportant and the decoherence term only survives. The nonunitary Wigner function evolution becomes

$$\dot{W}=\{H,W\}_{PB}+\sum_{n=1}^{\infty}\frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\times\frac{\partial^{2n+1}V}{\partial x^{2n+1}}\frac{\partial^{2n+1}W}{\partial p^{2n+1}}+D\frac{\partial^2 W}{\partial p^2}. \quad (2)$$

The decoherence term is in the form of a diffusive contribution to the dynamics with diffusion coefficient D. This is vital since it is the diffusion resulting from the opening of the system that limits the development of the fine structure in the momentum direction to a critical momentum scale σ_c. The time scale on which this process occurs is given by [12]

$$\tau_c \approx \frac{1}{\lambda}\ln\left(\frac{\sigma_p(0)}{\sigma_c}\right), \quad (3)$$

where $\sigma_p(0)$ is the initial width of a Gaussian wave packet in the momentum direction. Classical behavior is recovered, therefore, provided the environmentally induced diffusion process can prevent the development of fine structure, i.e., if

$$\tau_c \ll \tau_\hbar . \quad (4)$$

However, opening a system to a thermal environment has other consequences as well. In particular, the von Neumann entropy of the system, given by $S(t)=-\mathrm{Tr}\,\rho_r(t)\ln\rho_r(t)$, where $\rho_r(t)$ is the reduced density matrix of the system at time t, will increase; information is lost to the environment and initially pure, superposition states of the quantum system become classical mixtures in a very short time. Our ability to predict accurately the behavior of a classical system, which is exposed to a perturbation of the initial condition, depends very much on the nature of the dynamics. It is natural to ask whether this is true for the rate of information loss of its quantum analog when it is exposed to a perturbing environment.

As a first step towards answering this question (see also [15,16]) Zurek and co-workers have considered the inverted harmonic oscillator [17] of unit mass with Hamiltonian

$$H_S=\frac{p^2}{2}-\frac{\lambda^2x^2}{2}. \tag{5}$$

This is intended as a model of instability and, in fact, the dynamical behavior in phase space is dominated by a hyperbolic point at the origin. The unstable and stable directions and the rate at which initial phase space distributions expand and contract in these directions, respectively, are determined by λ. In this sense we call λ an instability parameter analagous to a Lyapunov exponent in a classical chaotic system. Indeed, at any point on a trajectory the sum of the Lyapunov exponents is zero. For a chaotic trajectory there must be one pair of nonzero Lyapunov exponents.

However, there are a number of reasons why we should question any conclusions drawn as to the implications for a real chaotic system based on so simple a model. First, there are no quantum corrections to the Wigner function evolution for this quadratic potential. The model does not allow for these influences on the dynamics, which, though small in the presence of an environment in comparison to the classical terms, nonetheless are generally *always present*. The stable and unstable manifolds associated with all hyperbolic points in Hamiltonian chaotic systems intersect both one another and those associated with other hyperbolic points [18]. In this way homoclinic and heteroclinic points are formed. The stable and unstable manifolds of the inverted oscillator intersect only at the hyperbolic origin in phase space. Clearly, therefore, the effect that the complicated distribution of homoclinic points might have on the open dynamics is not taken into account. Neither, of course, is the effect of heteroclinic points.

Notwithstanding these objections, however, the inverted oscillator remains a tractable model of instability both for a closed system and for an open system in the presence of an environment. As such, it deserves attention for the insights it might give regarding the qualitative and maybe quantitative behavior of genuine, open quantum analogs of classically chaotic systems.

The entropy production rate has been considered in the limit of high temperature and low dissipation. This entailed using the approximate Wigner function evolution in Eq. (2). Zurek and Paz show [12] that after a time determined by both λ and the strength of the interaction with the environment the rate of entropy increase approaches a constant

$$\dot{\mathcal{S}}\rightarrow\lambda, \tag{6}$$

i.e., the *quantum* entropy production rate is determined, in this approximation, by the *classical* instability parameter. Given that the classical Lyapunov exponent to which λ is analogous is equal to the Kolmogorov-Sinai (KS) entropy of the system [19], this is indeed a remarkable characterization [20]. It suggests that after a time, a quantum, classically chaotic system loses information to the environment at a rate determined *entirely* by the rate at which the classical system loses information as a result of its dynamics, namely, the KS entropy [21].

In this paper we will examine once more the ''toy'' model of Zurek and Paz. Apart from entropy production we will consider the experimentally relevant survival probability function. We will not be restricted to the assumption of low dissipation made by others [12]. As we will show, the asymptotic behavior of both the survival probability function and the rate of entropy increase will *not* be determined by λ alone but by λ in a specific combination with the dissipation parameter (which determines the strength of interaction with the environment). Our approach does not use the master equation approach of Zurek and Paz but rather Feynman-Vernon influence functional techniques [22]. This allows a straightforward analysis of strong coupling of the system to the environment.

The remainder of this paper is organized as follows. In Sec. II we define the initial state of both the system and the environment, determine the time evolution of the reduced density matrix of the system generally, and describe also how this allows us to calculate the von Neumann entropy $\mathcal{S}(t)$. We specify the nature of the environment more explicitly in Sec. III, leaving us in a position to consider finite-temperature evolution. In Sec. IV we shall define the survival probability function $P(t)$, calculate it for the inverted oscillator, and discuss its significance for quantum chaotic systems. In Sec. V we show analytically our generalization of Eq. (6) for the finite-temperature case. We state our conclusions in Sec. VI. Finally, the Appendix contains the more tedious details of the calculations.

II. REDUCED DENSITY MATRIX DYNAMICS

A. Initial state

We shall consider as our initial state the wave function

$$\psi(x_i,0)=(b\sqrt{\pi})^{-1/2}\exp\left(\frac{-(x_i-x_0)^2}{2b^2}+ip_0x_i\right), \tag{7}$$

for which it is easily verified that

$$\langle x_i\rangle=x_0,$$

$$\langle x_i^2\rangle=\frac{b^2}{2}+x_0^2,$$

$$\langle p\rangle=p_0,$$

$$\langle p^2\rangle=\frac{\hbar^2}{2b^2}+p_0^2,$$

$$(\Delta x_i)^2(\Delta p)^2=\frac{\hbar^2}{4},$$

i.e., a state of minimum uncertainty. However, we shall be concerned with density matrices and their evolution, so we define the normalized initial density matrix corresponding to $\psi(x_i,0)$ above by

$$\rho_S(x_i,x_i',0) \equiv \psi^*(x_i,0)\psi(x_i',0) = \mathcal{N}\exp[-\epsilon(x_i^2+x_i'^2)+ax_i+a^*x_i'], \quad (8)$$

where the normalization constant $\mathcal{N}$ is given by

$$\mathcal{N}=(b\sqrt{\pi})^{-1}\exp\left(-\frac{x_0^2}{b^2}\right), \quad (9)$$

with $\epsilon=(2b^2)^{-1}$ and $a=x_0/b^2+ip_0$.

We assume now that the system is put into contact with an environment at time $t=0$. We will use as our environmental model a bath of independent harmonic oscillators in thermal equilibrium at inverse temperature $1/k_BT$. This allows us to write the initial, uncorrelated state of the *combined* system in operator form as

$$\hat{\rho}_{SE}(0)=\hat{\rho}_S(0)\otimes\hat{\rho}_E(0), \quad (10)$$

where

$$\langle x_i|\hat{\rho}_S(0)|x_i'\rangle=\rho_S(x_i,x_i',0) \quad (11)$$

as defined in Eq. (8) and $\hat{\rho}_E(0)$ defines the thermal environment

$$\hat{\rho}_E(0)=\prod_n \left\{\left[1-\exp\left(-\frac{\hbar\omega_n}{k_BT}\right)\right] \times\sum_m \exp\left(-\frac{m\hbar\omega_n}{k_BT}\right)|m\rangle\langle m|\right\}, \quad (12)$$

i.e., in a factorized form because of the choice of noninteracting modes.

B. Time evolution propagator

The initial state being so defined, we now concentrate on the time evolution. We assume the Hamiltonian of the combined system to be

$$H_{SE}=H_S+H_E+H_I, \quad (13)$$

where H_S has been given in Eq. (5),

$$H_E=\sum_n \left(\frac{p_n^2}{2}+\frac{\omega_n^2q_n^2}{2}\right), \quad (14)$$

i.e., the Hamiltonian of our chosen bath with canonical commutation relations $[q_n,p_m]=i\hbar\delta_{n,m}$, and

$$H_I=-x\,c(t)\sum_n q_n, \quad (15)$$

the Hamiltonian of interaction describing the (possibly time-dependent) coupling of the inverted oscillator, through its position variable, to the position variable of each of the environmental oscillators.

The combined system and environment, initially in the pure product state of Eq. (10), will, of course, evolve unitarily under the action of the Hamiltonian H_{SE} of Eq. (13). We will be interested in the *reduced* density matrix of the system at some later time $t>0$, which we write as $\hat{\rho}_r(t)$. To arrive at this expression the path integral method of Feynman and Vernon is employed [13,22]. The initially uncorrelated state allows us to calculate the evolution kernel for the reduced density matrix in the position basis in the following way:

$$\rho_r(x_f,x_f',t)=\int dx_i\int dx_i' J_r(x_f,x_f',t|x_i,x_i',0)\rho_S(x_i,x_i',0), \quad (16)$$

where

$$J_r(x_f,x_f',t|x_i,x_i',0) = \int_{x_i}^{x_f}Dx\int_{x_i'}^{x_f'}Dx'\ \exp\left\{\frac{i}{\hbar}(S[x]-S[x'])\right\}\mathcal{F}[x,x'], \quad (17)$$

where S is the action of the system and $\mathcal{F}$ is the influence functional. Note that $\mathcal{F}=1$ in the absence of an environment. The propagator J_r has been calculated explicitly [23,24] by Koks, Matacz, and Hu and we quote here their result

$$J_r(x_f,x_f',t|x_i,x_i',0) = \frac{b_2}{2\pi\hbar}\exp\left[\frac{i}{\hbar}(b_1\Sigma_f\Delta_f-b_2\Sigma_f\Delta_i+b_3\Sigma_i\Delta_f-b_4\Sigma_i\Delta_i) -\frac{1}{\hbar}(a_{11}\Delta_i^2+a_{12}\Delta_f\Delta_i+a_{22}\Delta_f^2)\right], \quad (18)$$

where we have used the more convenient sum and difference coordinates defined by

$$\Delta\equiv x-x', \quad \Sigma\equiv(x+x')/2, \quad (19)$$

while $b_1,\ldots,b_4$ and a_{11}, a_{12}, and a_{22} are time-dependent terms defined entirely by the spectral density and temperature of the thermal environment. We will define them explicitly below when we specify the nature of the environment more precisely.

C. Reduced density matrix evolution

The final step in calculating the time evolution of the system is to use Eq. (18) to calculate the reduced density matrix $\rho_r(t)$ in the position basis via Eqs. (16) and (8). One finds, after some lengthy but trivial algebra,

$$\rho_r(x_f,x_f',t)=\frac{b_2\mathcal{N}}{\pi^2\sqrt{D}}\exp(-\Gamma_1\Delta_f^2-\Gamma_2\Delta_f\Sigma_f-\Gamma_3\Sigma_f^2 +\Gamma_5\Sigma_f+\Gamma_6\Delta_f+\Gamma_4), \quad (20)$$

where $\mathcal{N}$ has been defined in Eq. (9) above,

$$D=4\hbar^2\epsilon^2+b_4^2+8\epsilon\hbar a_{11}, \quad (21)$$

and we have made the abbreviations

$$\Gamma_1=\frac{a_{22}}{\hbar}+\frac{1}{D}\left\{\left(\frac{\epsilon}{2}+\frac{a_{11}}{\hbar}\right)b_3^2+\frac{b_4a_{12}b_3}{\hbar}-2\epsilon a_{12}^2\right\}, \quad (22)$$

$$\Gamma_2 = -2i\left\{\frac{b_1}{2\hbar} - \frac{1}{D}\left(\frac{b_2 b_3 b_4}{2\hbar} - 2b_2 a_{12}\epsilon\right)\right\}, \tag{23}$$

$$\Gamma_3 = \frac{2\epsilon b_2^2}{D}, \tag{24}$$

$$\Gamma_4 = a^{*2}A_7 + \frac{a^2\hbar^2}{D}(\epsilon - A_5) + \frac{2a_{11}\hbar a a^*}{D}, \tag{25}$$

$$\Gamma_5 = x_1\left(\frac{2a\hbar^2}{D}(\epsilon - A_5) + \frac{2a_{11}\hbar a^*}{D}\right) + y_1\left(2a^*A_7 + \frac{2a_{11}\hbar a}{D}\right), \tag{26}$$

and

$$\Gamma_6 = x_2\left(\frac{2a\hbar^2}{D}(\epsilon - A_5) + \frac{2a_{11}\hbar a^*}{D}\right) + y_2\left(2a^*A_7 + \frac{2a_{11}\hbar a}{D}\right), \tag{27}$$

but in which we have also defined

$$A_5 = \frac{ib_4}{2\hbar} - \frac{a_{11}}{\hbar}, \tag{28}$$

$$A_7 = \frac{1}{(\epsilon - A_5)}\left(\frac{1}{4} + \frac{a_{11}^2}{D}\right), \tag{29}$$

$$y_1 = x_1^* = \frac{ib_2}{\hbar}, \tag{30}$$

$$x_2 = \frac{ib_3}{2\hbar} - \frac{a_{12}}{\hbar}, \tag{31}$$

and

$$y_2 = \frac{ib_3}{2\hbar} + \frac{a_{12}}{\hbar}. \tag{32}$$

The expression in Eq. (20) is of a Gaussian form and can be diagonalized [25]. The von Neumann entropy

$$\mathcal{S}(t) = -\mathrm{tr}\ \rho_r(t)\ln\ \rho_r(t) \tag{33}$$

can therefore be calculated and written in the form

$$\mathcal{S}(t) = -\frac{1}{p_0}(p_0\ \ln\ p_0 + q\ \ln\ q), \tag{34}$$

where p_0 and q are defined by

$$p_0 = \frac{2\sqrt{\Gamma_3}}{\sqrt{\Gamma_1} + \sqrt{\Gamma_3}} \tag{35}$$

and

$$q = \frac{\sqrt{\Gamma_1} - \sqrt{\Gamma_3}}{\sqrt{\Gamma_1} + \sqrt{\Gamma_3}} = 1 - p_0, \tag{36}$$

using Eqs. (22) and (24) above.

III. ENVIRONMENT SPECIFICATION

A. Generalities

The influence functional used in the determination of the evolution kernel of Eq. (16) is determined entirely by the so-called dissipation and noise kernels of the chosen environment. If we now restrict each oscillator in the bath to have equal, unit mass we can write

$$\mu(s,s') = -\int_0^\infty d\omega\ I(\omega,s,s')\sin[\omega(s-s')] \tag{37}$$

and

$$\nu(s,s') = \int_0^\infty d\omega\ I(\omega,s,s')\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos[\omega(s-s')] \tag{38}$$

for the dissipation and noise kernels, respectively, where

$$\mathcal{F}[x,x'] = \exp\left\{-\frac{1}{\hbar}\int_0^t ds\int_0^s ds'\,\Delta(s) \times[\nu(s,s')\Delta(s') + 2i\mu(s,s')\Sigma(s')]\right\}. \tag{39}$$

Here $I(\omega,s,s')$ is called the *spectral density* of the environment as we have assumed the oscillators to have a continuous distribution of frequencies ω [13]. Notice that $\mu(s,s')$ is independent of the temperature of the environment.

Restricting the discussion now to the case of the inverted oscillator, we can determine the time-dependent quantities $b_1(t)$, $b_2(t)$, $b_3(t)$, and $b_4(t)$, as well as $a_{11}(t)$, $a_{12}(t)$, and $a_{22}(t)$, by

$$\begin{aligned} b_1(t) &= \dot{u}_2(t), \\ b_2(t) &= \dot{u}_2(0), \\ b_3(t) &= \dot{u}_1(t), \\ b_4(t) &= \dot{u}_1(0), \end{aligned} \tag{40}$$

and

$$a_{ij} = \frac{1}{1+\delta_{ij}}\int_0^t ds\int_0^t ds'\,v_i(s)\nu(s,s')v_j(s'). \tag{41}$$

The functions u_i and v_i that determine these quantities are solutions of the differential equations

$$\ddot{u}(s) - \lambda^2 u(s) + 2\int_0^s ds'\,\mu(s,s')u(s') = 0, \tag{42}$$

$$\ddot{v}(s)-\lambda^2 v(s)-2\int_s^t ds'\mu(s,s')v(s')=0 \tag{43}$$

when the boundary conditions

$$u_1(0)=v_1(0)=1, \quad u_1(t)=v_1(t)=0,$$
$$u_2(0)=v_2(0)=0, \quad u_2(t)=v_2(t)=1$$

are imposed.

B. Calculation of $b_i(t)$

It is clear from Eqs. (40) and (42) above that the quantities $b_1(t)$–$b_4(t)$ depend only the dissipation kernel $\mu(s,s')$ and not on the temperature of the environment. We will concentrate on spectral densities of the form

$$I(\omega,s,s')=\frac{2\gamma_0}{\pi}\omega \exp\left(-\frac{\omega}{\omega_c}\right)c(s)c(s'), \tag{44}$$

i.e., densities of an *Ohmic* type with an upper or cutoff frequency ω_c [13]. Using Eq. (44) in Eq. (37) we find

$$\mu(s,s')\to 2\gamma_0 c(s)c(s')\delta'(s-s') \tag{45}$$

as $\omega_c\to\infty$. A further restriction to the case of constant coupling constants, i.e., $c(t)=1\ \forall\ t$, enables us to write for the inverted oscillator

$$\begin{aligned} b_1(t)&=(-\gamma_0+\kappa \coth \kappa t),\\ b_4(t)&=(-\gamma_0-\kappa \coth \kappa t),\\ b_2(t)&=\frac{\kappa \exp(\gamma_0 t)}{\sinh \kappa t},\\ b_3(t)&=\frac{-\kappa \exp(-\gamma_0 t)}{\sinh \kappa t}. \end{aligned} \tag{46}$$

The shorthand definition

$$\kappa=\sqrt{\lambda^2+\gamma_0^2} \tag{47}$$

has been used here and we will see that it is an important quantity in the sections below. The asymptotic behavior of b_i, $i=1,\ldots,4$, in Eq. (46) for large κt can easily be determined and each is given by one of

$$b_1(t)\to(-\gamma_0+\kappa), \tag{48}$$

$$b_2(t)\to 2\kappa \exp[-(\kappa-\gamma_0)t], \tag{49}$$

$$b_3(t)\to -2\kappa \exp[-(\kappa+\gamma_0)t], \tag{50}$$

or

$$b_4(t)\to(-\gamma_0-\kappa). \tag{51}$$

Finally, $v_1(t)$ and $v_2(t)$ are required for the calculation of each a_{ij} in Eq. (41). Again, for the inverted harmonic oscillator they are given by

$$v_1(s)=\frac{\sinh[\kappa(t-s)]\exp(\gamma_0 s)}{\sinh(\kappa t)} \tag{52}$$

and

$$v_2(s)=\frac{\sinh(\kappa s)\exp[\gamma_0(s-t)]}{\sinh(\kappa t)} \tag{53}$$

for $s\in[0,t]$, where t is the time at which we wish to calculate the reduced density matrix.

C. Calculation of the finite temperature $a_i(t)$ coefficients

To examine the asymptotic rate of entropy production of the inverted oscillator, in an environment at a *finite* temperature, we must, of course, calculate the appropriate a_{ij} coefficients. This will allow to analytically derive the asymptotic rate of von Neumann entropy increase. The lengthy details are relegated to the Appendix. Writing $\hat{\lambda}:=\lambda/\kappa$ and $z:=\kappa t$, we find eventually

$$a_{11}(t,0)=\frac{\hat{\gamma}_0}{(2\hat{\lambda}\sinh z)^2}\times\sum_{n=0}^{\infty} f(n)\left(\frac{d_1(n)f_1(z)}{\hat{\gamma}_0}+d_2(n)f_2(z)\right), \tag{54}$$

$$a_{12}(t,0)=\frac{e^{-\hat{\gamma}_0 z}}{2(\hat{\lambda}\sinh z)^2}\times\sum_{n=0}^{\infty} f(n)[d_1(n)f_3(z)-d_2(n)f_4(z)], \tag{55}$$

$$a_{22}(t,0)=\frac{e^{-2\hat{\gamma}_0 z}}{(2\hat{\lambda}\sinh z)^2}\times\sum_{n=0}^{\infty} f(n)[d_1(n)f_5(z)+d_2(n)f_6(z)], \tag{56}$$

where the functions $f_i(z), i=1,\ldots,6$, are defined in the Appendix, as are $d_1(n)$ and $d_2(n)$. Also given in the Appendix is the asymptotic behavior of each (finite-temperature) a_{ij}.

IV. SURVIVAL PROBABILITY FUNCTION

A. Definition and context

In this section we will consider the *survival probability function* $P(t)$, defined by

$$P(t)=\mathrm{Tr}[\rho(0)\rho(t)] \tag{57}$$

for a system initially described by the density matrix $\rho(0)$. In particular, we will be interested in the asymptotic behavior.

The function $P(t)$ has been examined before in the context of quantum chaos. Tameshtit and Sipe [27] have considered its behavior in time for both regular and chaotic systems coupled in a nondemolition fashion to a high-temperature thermal reservoir. Using a master equation approach and random matrix theory they show that when suitably averaged, the behavior of $P(t)$ displays substantial differences depending on the nature of the underlying dynamics. This function has also been examined theoretically for pure states [28,29], even for the inverted oscillator model [17]. However, the fact that it is accessible experimentally [30] makes it a particularly important quantity to examine.

For *pure* initial states $\rho(0)=|\phi(0)\rangle\langle\phi(0)|$ and unitary evolution $|\phi(t)\rangle=\hat{U}(t)|\phi(0)\rangle$ we have

$$P(t)=|\langle\phi(0)|\phi(t)\rangle|^2, \tag{58}$$

i.e., the probability of the system being in the initial state at a later time t. $P(t)$, as defined in Eq. (57) above, is a generalization of this applicable to systems that may not be initially pure and/or for which the evolution in time is nonunitary.

B. Survival probability for the open inverted oscillator

We will now examine the effect of the thermal bath on the survival probability function of the inverted harmonic oscillator. For simplicity we will consider our initial state to be centered at the origin of phase space, i.e., we take our initial wave function to be given by Eq. (7) with $x_0=p_0=0$. The initial density matrix is then given by Eq. (8) with $a=0$ and $\mathcal{N}=(b\sqrt{\pi})^{-1}$. These initial conditions simplify the form of the reduced density matrix considerably, leading to Eq. (20) with $\Gamma_4=\Gamma_5=\Gamma_6=0$. The survival probability function can now be simply calculated according to Eq. (57) above. We find

$$\begin{aligned}P(t)&=\mathrm{Tr}[\rho_S(0)\rho_r(t)]\\&=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\langle x|\rho_S(0)|x'\rangle\langle x'|\rho_r(t)|x\rangle\\&=\frac{1}{b\pi^2}\left\{\frac{\Gamma_3}{(\epsilon+2\Gamma_1)(\epsilon+\Gamma_3/2)-\Gamma_2^2/4}\right\}^{1/2},\end{aligned} \tag{59}$$

where we have yet to specify the magnitude of the parameters of the environment. For the finite-temperature case, we can easily determine the asymptotic behavior. Using the asymptotic behavior of each $b_i(t)$ [given by Eqs. (48)–(51)], the asymptotic behavior of each $a_{ij}(t,0)$ [given in Eqs. (A18)–(A20)], and the definitions of each Γ_i [Eqs. (22)–(24)] we easily see that for a fixed temperature

$$\{(\epsilon+2\Gamma_1)(\epsilon+\Gamma_3/2)-\Gamma_2^2/4\}^{1/2}\to C_1 \tag{60}$$

for large times, where C_1 is a (temperature-dependent) constant. Consequently, the large time behavior of $P(t)$ is determined *entirely* by that of Γ_3. As $t(z)\to\infty$ we see from Eqs. (51), (A18), and (21) that D goes to the (temperature-dependent) constant

$$D_{asym}=4\hbar^2\epsilon^2+(\gamma_0+\kappa)^2-(4\epsilon\hbar\gamma_0)\coth\left(\frac{(\gamma_0-\kappa)\hbar}{2k_BT}\right). \tag{61}$$

Now, using its definition in Eq. (24), the asymptotic behavior of b_2 in Eq. (49), and that of D in Eq. (61) we find, finally,

$$P(t)\underset{t\to\infty}{\sim}C_2\exp[-(\kappa-\gamma_0)t], \tag{62}$$

where C_2 is another temperature-dependent constant. We have checked the accuracy of this result numerically by calculating $\ln[P(t-1)/P(t)]$, which, if Eq. (62) is to be believed, has $\kappa-\gamma_0\equiv\kappa'=\sqrt{\lambda^2+\gamma_0^2}-\gamma_0$ as its asymptotic value. Excellent agreement was found. This is a generalization of the result of Heller [6] to the *open* quantum inverted oscillator and reduces to it as $\gamma_0\to 0$, i.e., as the system interacts more weakly with the environment, as required. It would be natural to conjecture in the spirit of Zurek and Paz [12] that the behavior in Eq. (62) might be expected to hold for quantizations of classically chaotic systems in interaction with an environment.

V. ASYMPTOTIC RATE OF ENTROPY INCREASE: ANALYTICAL RESULTS

We are now in a position to derive the rate at which the von Neumann entropy will increase when $z=\kappa t$ is large, i.e., at long times and/or when the Lyapunov exponent is large, for the finite temperature case. We can rewrite Eq. (35) as

$$p_0=\frac{2\alpha}{1+\alpha}, \tag{63}$$

where we have made the abbreviation $\alpha:=\sqrt{\Gamma_3/\Gamma_1}$. Also, Eq. (34) gives

$$\begin{aligned}\frac{dS}{dt}&=\frac{d}{dt}\left\{-\frac{1}{p_0}[p_0\ln p_0+(1-p_0)\ln(1-p_0)]\right\}\\&=\cdots=\frac{\dot{p}_0}{p_0^2}\ln(1-p_0),\end{aligned} \tag{64}$$

with the overdot denoting a derivative with respect to t. Combining Eqs. (63) and (64), we find, *for all t*,

$$\frac{dS}{dt}=\frac{\dot{\alpha}}{2\alpha^2}\ln\left\{\frac{1-\alpha}{1+\alpha}\right\}. \tag{65}$$

Therefore, both the entropy $S(t)$ and its rate of change dS/dt are determined entirely by the time-dependent coefficients Γ_1 and Γ_3 that arise in the expression for the reduced density matrix (20). [Note that these are both independent of the center of the initial minimum uncertainty wave packet (x_0,p_0).]

The asymptotic expression for D given by Eq. (61), along with that of Eq. (49), gives the asymptotic behavior of Γ_3 defined in Eq. (24):

$$\lim_{\kappa t\to\infty}\Gamma_3=\frac{8\epsilon\kappa^2}{D_{asym}}\exp[-2(\kappa-\gamma_0)t]. \tag{66}$$

Inspection of the asymptotics of the various environmental terms used to define Γ_1 in Eq. (22), in particular Eqs. (A20), (61), (A18), (50), (51), and (A19), give

$$\lim_{\kappa t\to\infty}\Gamma_1=\frac{a_{22}(t\to\infty,0)}{\hbar}=\frac{\gamma_0}{2\hbar}\coth\left(\frac{(\gamma_0+\kappa)\hbar}{2k_BT}\right),\quad(67)$$

i.e., also a (temperature-dependent) constant.

The asymptotic behavior of α can now be determined using Eqs. (66) and (67):

$$\lim_{\kappa t\to\infty}\alpha=\xi\,\exp[-(\kappa-\gamma_0)t],\quad(68)$$

where ξ is a temperature-dependent, but time-independent constant. Clearly, $\alpha\to 0$ as $t\to\infty$, which means that $p_0\to 0$ too. Considering Eq. (64) as $t\to\infty$ we find

$$\frac{dS}{dt}=\frac{\dot{p}_0}{p_0^2}(-p_0+p_0^2/2+\cdots)\approx-\frac{\dot{p}_0}{p_0}=\frac{-\dot{\alpha}}{\alpha(1+\alpha)},\quad(69)$$

where we have used Eq. (63) in the last step. Finally, if we use the asymptotic expression for α, Eq. (68), we find

$$\frac{dS}{dt}\underset{\kappa t\to\infty}{\sim}\kappa-\gamma_0=\kappa'.\quad(70)$$

This result gives the asymptotic rate of entropy increase in situations where energy dissipation is important and cannot be neglected. As γ_0 determines the strength of the coupling to the environment, we can now see that the rate at which an unstable, possibly chaotic system will lose information to the environment *does* depend on the coupling strength [26]. Note, however, that this asymptotic rate reduces to the asymptotic rate found previously in the weakly coupled regime

$$\frac{dS}{dt}\underset{\kappa t\to\infty}{\sim}\sqrt{\lambda^2+\gamma_0^2}-\gamma_0\approx\lambda\quad\text{when}\quad\gamma_0\ll\lambda,\quad(71)$$

as required.

VI. CONCLUSION

In this paper we have used the inverted harmonic oscillator to model instability in open quantum systems. We have found that both the survival probability function and the von Neumann entropy increase depend, for large times, on the degree of instability in the system *and* on the strength of interaction with the environment in a simple way. The purpose of studying such an elementary system is to build up some degree of intuition as to the behavior of quantum chaotic systems coupled to an environment. *A priori* the claims of applicability of an inverted oscillator to modeling a chaotic system should be treated with caution. We will now discuss to what degree the results for the oscillator can serve as a guide to actual quantum behavior in chaotic systems.

With regard to the survival probability function prediction we mention the study by D'Ariano *et al.* [31] of classical and quantum structures in the kicked-top model. No environment was included in their model, but they have shown that the survival probability function (autocorrelation function in their paper) provides an excellent means with which to compare classical and quantum invariant structures (periodic points), at least for predominantly regular kicked tops. This situation changes when most of the tori have been destroyed. We are currently examining the influence a perturbed evolution has on this correspondence. Practically, the survival probability function can be obtained from an experimental spectrum (see the comments by Heller in [6] and also [30]) and is a useful concept in studies of molecular spectra. The longest events in the time domain determine the broadening of the peaks in the spectrum for an unstable periodic orbit, with the width being given by the classical Lyapunov exponent. Our result, Eq. (62), suggests that this may change in the presence of an environment due to γ_0.

For the entropy production prediction we mention two studies of the quantum kicked rotor model [1,2,7] evolving as it interacts with a perturbing environment [20,32]. In the first [20], the constant von Neumann entropy production rate was calculated for various values of the nonlinearity parameter K of the system. The Lyapunov exponent λ can be approximated by $\lambda\approx\ln(K/2)$ for large K and a *linear* dependence of the constant entropy production rate on the Lyapunov exponent was found, i.e.,

$$\dot{S}=a\lambda+b,\quad(72)$$

where a and b are constants determined by the environment. Moreover, this behavior is seen in the mixed phase spaces resulting from low values of K provided one considers only *local* Lyapunov exponents to be relevant. In the second study [32] an initial state is perturbed unitarily as it evolves in time with the possible perturbations at each time step being taken from an ensemble. Averaging over the possible realizations requires that the state of the system be described by a density matrix as it is mixed. In this way the entropy can increase. It is shown that the eventually constant rate at which an initial coherent state produces entropy depends to a great degree upon where in phase space the center of the wave packet is situated: The more chaotic the area of initial localization, as quantified by a locally averaged Lyapunov exponent, the larger the constant entropy production rate.

We conclude, therefore, that there is indeed some value in using the toy model considered in this paper and others [12] as a model of instability in open quantum systems. The conjectures that follow from it should be tested in more systems that are classically chaotic.

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APPENDIX

In this appendix we will give the explicit definitions of the functions f_i, $i=1,\ldots,6$, $d_1(n)$, and $d_2(n)$ used to write the finite-temperature expressions for each a_{ij} in Eqs. (54)–(56). We will also derive their large-time limits (A18)–(A20).

Once again we will choose constant coupling constants in the Ohmic spectral density of Eq. (44) and we will also assume a large but finite ω_c. This allows us to write

$$I(\omega,s,s')\approx\frac{2\gamma_0\omega}{\pi}.\quad(A1)$$

To calculate the finite-temperature noise kernel of Eq. (38) we will now *formally* expand the coth function in a power series [33] and integrate term by term. Formally then

$$\nu(s,s')=\frac{2\gamma_0}{\pi}\sum_{n=0}^{\infty}\frac{2^{2n}B_{2n}}{(2n)!}\left(\frac{\hbar}{2k_BT}\right)^{2n-1}\times\int_0^{\infty}d\omega\;\omega^{2n}\cos\omega(s-s'). \tag{A2}$$

B_{2n} are Bernoulli numbers. Defining

$$f(n):=\frac{2^{2n}B_{2n}}{(2n)!}\left(\frac{\hbar}{2k_BT}\right)^{2n-1}, \tag{A3}$$

we can rewrite this formal expansion as

$$\nu(s,s')=2\gamma_0\sum_{n=0}^{\infty}f(n)\delta^{(2n)}(s-s'), \tag{A4}$$

i.e., as an infinite sum of derivatives of the δ function. This expression can now be used in Eq. (41), with Eqs. (52) and (53), to calculate each a_{ij}. Writing $\hat{\lambda}:=\lambda/\kappa$, $\hat{\gamma}_0:=\gamma_0/\kappa$, and $z:=\kappa t$, we find

$$a_{11}(t,0)=\frac{\hat{\gamma}_0}{(2\hat{\lambda}\sinh z)^2}\times\sum_{n=0}^{\infty}f(n)\left(\frac{d_1(n)f_1(z)}{\hat{\gamma}_0}+d_2(n)f_2(z)\right), \tag{A5}$$

$$a_{12}(t,0)=\frac{e^{-\hat{\gamma}_0 z}}{2(\hat{\lambda}\sinh z)^2}\times\sum_{n=0}^{\infty}f(n)[d_1(n)f_3(z)-d_2(n)f_4(z)], \tag{A6}$$

$$a_{22}(t,0)=\frac{e^{-2\hat{\gamma}_0 z}}{(2\hat{\lambda}\sinh z)^2}\times\sum_{n=0}^{\infty}f(n)[d_1(n)f_5(z)+d_2(n)f_6(z)], \tag{A7}$$

where

$$d_{\{1 \atop 2\}}(n):=\frac{\kappa^{2n}}{2}[(\hat{\gamma}_0-1)^{2n}\pm(\hat{\gamma}_0+1)^{2n}] \tag{A8}$$

and

$$f_1(z):=\hat{\gamma}_0^2\cosh 2z+\hat{\gamma}_0\sinh 2z-\exp(2\hat{\gamma}_0 z)+1-\hat{\gamma}_0^2, \tag{A9}$$

$$f_2(z):=\hat{\gamma}_0\sinh 2z+\cosh 2z-\exp(2\hat{\gamma}_0 z), \tag{A10}$$

$$f_3(z):=\cosh z[\exp(2\hat{\gamma}_0 z)-1]-\hat{\gamma}_0\sinh z[\exp(2\hat{\gamma}_0 z)+1], \tag{A11}$$

$$f_4(z):=\hat{\gamma}_0\cosh z[1-\exp(2\hat{\gamma}_0 z)]+\sinh z[\exp(2\hat{\gamma}_0 z)-1]+2\hat{\gamma}_0^2\sinh z, \tag{A12}$$

$$f_5(z):=1-\hat{\lambda}^2\exp(2\hat{\gamma}_0 z)-\hat{\gamma}_0\exp(2\hat{\gamma}_0 z)\times(\hat{\gamma}_0\cosh 2z-\sinh 2z), \tag{A13}$$

$$f_6(z):=\hat{\gamma}_0[\exp(2\hat{\gamma}_0 z)(\hat{\gamma}_0\sinh 2z-\cosh 2z)+1]. \tag{A14}$$

Asymptotics of a_{ij}

It is easy to show that

$$\frac{f_1(z)}{\sinh^2 z}\underset{z\to\infty}{\to}2(\hat{\gamma}_0^2+\hat{\gamma}_0) \tag{A15}$$

and

$$\frac{f_2(z)}{\sinh^2 z}\underset{z\to\infty}{\to}2(\hat{\gamma}_0+1). \tag{A16}$$

These expressions allow us to determine the asymptotic behavior of $a_{11}(t,0)$ defined in Eq. (54). We find

$$\lim_{t\to\infty}a_{11}(t,0)=\frac{\hat{\gamma}_0}{4\hat{\lambda}^2}\sum_{n=0}^{\infty}f(n)\left(\frac{d_1(n)}{\hat{\gamma}_0}[2(\hat{\gamma}_0^2+\hat{\gamma}_0)]+d_2(n)[2(\hat{\gamma}_0+1)]\right)=\frac{\hat{\gamma}_0(\hat{\gamma}_0+1)}{2\hat{\lambda}^2}\sum_{n=0}^{\infty}f(n)(\hat{\gamma}_0-1)^{2n}\kappa^{2n}. \tag{A17}$$

One next uses the definition of $f(n)$ in Eq. (A3) and the formal power series expansion of the coth function to derive the desired expression given in Eq. (A18) below. To derive Eqs. (A19) and (A20) we proceed along similar lines:

$$a_{11}(t,0)\to-\frac{\hat{\gamma}_0\kappa}{2}\coth\left(\frac{(\gamma_0-\kappa)\hbar}{2k_BT}\right), \tag{A18}$$

$$a_{12}(t,0)\to\kappa e^{-(1-\hat{\gamma}_0)z}\coth\left(\frac{(\gamma_0+\kappa)\hbar}{2k_BT}\right), \tag{A19}$$

$$a_{22}(t,0)\to\frac{\hat{\gamma}_0\kappa}{2}\coth\left(\frac{(\gamma_0+\kappa)\hbar}{2k_BT}\right). \tag{A20}$$

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